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Congruences of Sheffer stroke basic algebras

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Abstract

In this paper, congruences and (order, prime and compatible) filters are introduced in Sheffer stroke basic algebras. They are interrelated with each other. Also some relationships between filters and congruences by focusing on their properties in these structures are given. In addition to these, a basic algebra is constructed by the help of compatible filter and $\mathcal{A}/\Theta(F)$. Moreover, a representation theorem for Sheffer stroke basic algebra by using prime filters and $\mathcal{A}/\Theta(F)$ is stated and proved.

1 Introduction

When a structure is being constructed as a mathematical model, the first thing to do is always to get rid of unnecessary expressions. For this reason, mathematicians attempt to express equivalent statements as possible as with the least number of axioms or the least number of operations and so on. For instance, Tarski dealt with a problem which was about the least number of axioms for Abelian groups and he shown that Abelian groups can be characterized from the point of divisor operator view via a single axiom $a/(b/(c/(a/b))) = c$ [20]. Taking this reduction into account, we consider Sheffer stroke operation for algebraic structures. Firstly defining the Sheffer stroke operation in 1913 [19], H. M. Sheffer indicated that all Boolean functions could be expressed by means of Sheffer stroke operation. McCune et

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al. axiomatized Boolean algebras only by the Sheffer stroke operation in 2002 [17]. This operation has a crucial role for computer systems. To put it more explicitly, it has an useful application in chip technology as it allows to have all diodes on the chip forming processor in a computer and also in a uniform manner. Hence, this is cheaper and simpler than to use different diodes for other logical connectives such as conjunction, disjunction, negation and etc.

Initially, implication reduct attempted about connectives of Boolean algebras was given by J. C. Abbott under the name of *implication algebra* [1]. Since the logic for quantum mechanics was explained by the help of orthomodular lattices, Abbott attained implication reducts of orthomodular lattices, so-called *orthoimplication algebras* in [2]. Afterwards, this attempt was generalized to implication reducts of ortholattices by Chajda and Halaš [3], Chajda [4] and it was also generalized to orthomodular lattices by Chajda, Halaš and Länger without the compatibility condition [5]. On the other hand, the operation reducts have extensively studied in the recent times. The most accentuated operation is Sheffer stroke operation among these studies including various operator. The latest example of this attempt was given for non-associative MV-algebras by Chajda et al. [6].

Basic algebras were given by Chajda and Emanovský [7], see also Chajda [8] and Chajda et al. [9] for more information. In accordance with these studies, Oner and Senturk introduced a reduction of basic algebras by means of only Sheffer stroke operation [18]. Basic algebras are widely used in different non-classical logics because they do not only include orthomodular lattices $\mathcal{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ where $\neg x = x^{\perp}$ and $x \oplus y = (x \wedge y^{\perp}) \vee y$ but also yield an axiomatization of the logic of quantum mechanics along with MV-algebras [15], which is obtained an axiomatization of many-valued Łukasiewicz logics; see Chajda [10] and Chajda et al.[11]. After then, Chajda and Kühr defined an internal characterization of congruence kernels of basic algebras and attained a finite basis of ideal terms [12].

In the present paper, the fundamental target of our research is to define (order, prime and compatible) filters and congruences of Sheffer stroke basic algebras and determine the relationships with each other. We briefly mention appearance of the concept of Sheffer stroke basic algebras from basic algebras. And, we refer certain results obtained by several researchers regarding the concept of filters and congruences of algebraic structures in Section 1. In Section 2, we recall fundamental definitions and notations with reference to Sheffer stroke basic algebras. In Section 3, we firstly define filters and congruences of Sheffer stroke basic algebra and we establish a representation theorem for Sheffer stroke basic algebra by using prime filters and $\mathcal{A}/\Theta(F)$. In the sequel, we indicate that each congruence in this structure is uniquely determined by its filter $[1]_{\Theta}$ and so to characterize congruences, which is adequate to describe filters. Finally, we construct a bridge between congruences and filters of Sheffer stroke basic algebras. In Section 4, we briefly summarize the findings obtained throughout this paper.

2 Preliminaries

The fundamental concepts that are needed throughout the paper are given in this section.

Definition 2.1. *[14] An algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called a basic algebra, if it verifies the following axioms:*
$(BA1)$ $a \oplus 0 = a$,
$(BA2)$ $\neg\neg a = a$,
$(BA3)$ $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$,
$(BA4)$ $\neg(\neg(\neg(\neg a \oplus b) \oplus b) \oplus c) \oplus (a \oplus c) = \neg 0$.

Definition 2.2. *[13] Let $\mathcal{A} = (A, |)$ be a groupoid. If the following conditions are satisfied, then the operation $| : A \times A \to A$ is called a Sheffer stroke operation.*
$(S1)$ $a|b = b|a$,
$(S2)$ $(a|a)|(a|b) = a$,
$(S3)$ $a|((b|c)|(b|c)) = ((a|b)|(a|b))|c$,
$(S4)$ $(a|((a|a)|(b|b)))|(a|((a|a)|(b|b))) = a$.

If also the following identity

$(S5)$ $b|(a|(a|a)) = b|b$,

is satisfied, then it is said to be an ortho-Sheffer stroke operation.

Lemma 2.3. *[13] Assume that $\mathcal{A} = (A, |)$ is a groupoid. The binary relation $\leq$ defined on A as below*

$$a \leq b \text{ if and only if } a|b = a|a$$

is a partial order on A.

Lemma 2.4. *[13] Assume that $|$ is a Sheffer stroke operation on A and $\leq$ is the induced order of $\mathcal{A} = (A, |)$. Then we have*

(i) $a \leq b$ *if and only if* $b|b \leq a|a$,

(ii) $a|(b|(a|a)) = a|a$ *is the identity of* $\mathcal{A}$,

(iii) $a \leq b$ *implies* $b|c \leq a|c$, *for all* $c \in A$,

(iv) $x \leq a$ *and* $x \leq b$ *imply* $a|b \leq x|x$.

Definition 2.5. *[18] An algebra* $(A, |)$ *of type* $< 2 >$ *is called a Sheffer stroke basic algebra if it satisfies the following identities:*
$(SH1)$ $(a|(a|a))|(a|a) = a$,
$(SH2)$ $(a|(b|b))|(b|b) = (b|(a|a))|(a|a)$,
$(SH3)$ $(((a|(b|b))|(b|b))|(c|c))|((a|(c|c))|(a|(c|c))) = a|(a|a)$.

We have some simple properties for *Sheffer Stroke basic algebras*. Furthermore, such an algebraic structure has a constant 1 as it is the case for implication basic algebras [14].

Lemma 2.6. *[18] Assume that* $(A, |)$ *is a Sheffer Stroke basic algebra. Then the element* 1 *is a constant in* A *and the structure* $(A, |)$ *provides the following identities:*

(i) $a|(a|a) = 1$,

(ii) $a|(1|1) = 1$,

(iii) $1|(a|a) = a$,

(iv) $((a|(b|b))|(b|b))|(b|b) = a|(b|b)$,

(v) $(b|(a|(b|b)))|(a|(b|b)) = 1$.

For revealing the structure of Sheffer stroke basic algebras, we give a partial order on A by

Lemma 2.7. *[18] Assume that* $(A; |)$ *is a Sheffer stroke basic algebra. A binary relation* $\leq$ *is defined on* A *as follows:*

$$a \leq b \quad \textit{if and only if} \quad a|(b|b) = 1.$$

Then the binary relation $\leq$ *is a partial order on* A *such that* $a \leq 1$, *for each* $a \in A$. *Moreover, we have that*

$$c \leq (a|(c|c)) \quad \textit{and} \quad a \leq b \quad \textit{implies} \quad b|(c|c) \leq a|(c|c)$$

for all $a, b, c \in A$.

The partial order $\leq$ given in the Lemma 2.7 is called *induced partial order* of Sheffer stroke basic algebra $(A; |)$. There is a bridge between Sheffer stroke basic algebras and lattice structures which are given in [18] as following theorem and corollary.

Theorem 2.8. *[18] Suppose that $(A; |)$ is a Sheffer stroke basic algebra and let $\leq$ be its induced partial order. Then $(A; \leq)$ is a join semi-lattice with the greatest element 1 where $a \vee b = (a|(b|b))|(b|b)$.*

Corollary 2.9. *[18] Let $(A; |)$ be a Sheffer Stroke basic algebra with the least element 0 and the greatest element 1. Then $(A; \vee, \wedge, \neg, 0, 1)$ is a lattice with an antitone involution.*

3 Filters of Sheffer stroke basic algebras

It is known that almost all algebraic structures corresponding to some logics are weakly regulars. In other words, each congruence of these algebraic structures is uniquely identified by its kernel $[1]_\Phi$. If we define any two congruences like Φ and Θ of these algebraic structures such that $[1]_\Phi = [1]_\Theta$, then we get $\Phi = \Theta$. So, we have to define the kernel for describing a congruence of Sheffer Stroke basic algebra. For this aim, we give the following definition.

Throughout this paper, the algebraic structure $\mathcal{A} = (A; |)$ represents a Sheffer stroke basic algebra.

Definition 3.1. *Let F be a subset of A with $1 \in F$. Then, F is called a filter of $\mathcal{A}$ if it satisfies the following conditions, for all $a, b, c \in A$:*

$(F_{SH}1)$ *If $a \in F$ and $a|(b|b) \in F$, then $b \in F$,*
$(F_{SH}2)$ *If $a|(b|b) \in F$ with $b|(a|a) = 1$, then $(b|(c|c))|((a|(c|c))|(a|(c|c))) \in F$.*

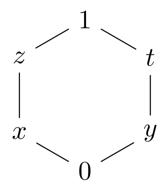
Moreover, when F verifies the following condition

$(F_{SH}3)$ *If $a|(b|b) \in F$ and $b|(a|a) \in F$, then $(c|(a|a))|((c|(b|b))|(c|(b|b))) \in F$,*

it is called a compatible filter of $\mathcal{A}$.

The set of all filters and all compatible filters of $\mathcal{A}$ are denoted by $\mathbf{F}(\mathcal{A})$ and $\mathbf{CF}(\mathcal{A})$, respectively.

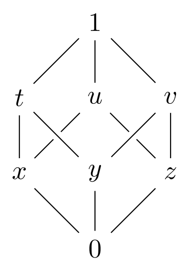
Example 3.2. *Let $A = \{0, x, y, z, t, 1\}$, where $0 < x < z < 1$ and $0 < y < t < 1$ but x, y, t and z, y, t are incomparable. The relations of elements in A are given and the operation $|$ on this structure is defined as the following figure and the table, respectively:*



$\vert$	0	x	y	z	t	1
0	1	1	1	1	1	1
x	1	t	1	t	1	t
y	1	1	z	1	z	z
z	1	t	1	y	1	y
t	1	1	z	1	x	x
1	1	t	z	y	x	0

Then $\mathcal{A} = (A; |)$ is a Sheffer stroke basic algebra. Since $F = \{x, z, 1\}$ satisfies the conditions of Definition 3.1, it is a compatible filter of $\mathcal{A}$.

Example 3.3. *Let $B = \{0, x, y, z, t, u, v, 1\}$. The relations of elements in B are given and the operation $|$ on this structure is defined as the following figure and the table, respectively:*



$\vert$	0	x	y	z	t	u	v	1
0	1	1	1	1	1	1	1	1
x	1	v	v	1	v	v	1	v
y	1	v	u	1	u	1	u	u
z	1	1	1	t	1	t	t	t
t	1	v	u	1	z	v	u	z
u	1	v	1	t	v	y	t	y
v	1	1	u	t	u	t	x	x
1	1	v	u	t	z	y	x	0

Then $\mathcal{B} = (B; |)$ is a Sheffer stroke basic algebra. $Q = \{1, t\}$ is a filter of $\mathcal{B}$, but it is not a compatible filter of $\mathcal{B}$, since $0|(z|z) = 1 \in Q$ and $z|(0|0) = t \in Q$. On the other hand, for all $k \in B$, we have

$$(k|(0|0))|((k|(z|z))|(k|(z|z))) = (k|1)|((k|t)|(k|t)).$$

Substuting $[k := u]$ gives us

$$(u|1)|((u|t)|(u|t)) = y|(v|v) = y|x = v \notin Q.$$

Therefore, the filter Q does not satisfy the condition $(F_{SH}3)$.

Definition 3.4. *If the following condition*

$$a|(b|c) = (a|b)|c$$

is satisfied for each $a, b, c \in A$, then $\mathcal{A}$ is called an associative Sheffer stroke basic algebra.

Definition 3.5. *Let U be a subset of A. The subset U is an order-filter if it satisfies the condition*

$$a \in U \quad \text{and} \quad a|(b|b) = 1 \implies b \in U.$$

Example 3.6. *Let $\mathcal{A} = (A; |)$ be a Sheffer stroke basic algebra given in Example 3.3. When we take $U = \{x, t, 1\}$, then U is an order-filter of $\mathcal{B}$.*

Before giving the following lemma, we define the operation $\circledast$ on A as

$$a \circledast b = (a|b)|(a|b).$$

Lemma 3.7. *Let $F \subseteq A$ with $1 \in F$. Then*

(i) If $F \in \mathbf{F}(\mathcal{A})$ and $\leq$ is the induced order of $\mathcal{A}$, then F is an order-filter of $(A; \leq)$. But the converse is not true,

(ii) F satisfies condition $(F_{SH}1)$ if and only if F is an order-filter closed under the operation $\circledast$,

(iii) If $\mathcal{A}$ is associative, then F satisfies only condition $(F_{SH}1)$ if and only if $F \in \mathbf{CF}(\mathcal{A})$.

Proof. (i) Let $F \in \mathbf{F}(\mathcal{A})$ and $\leq$ be the induced order of $\mathcal{A}$. It is clear that F is an order-filter by Lemma 2.7 and Definition 3.5.

Let $U = \{x, t, 1\}$ be defined as Example 3.6. U is an order-filter but it does not satisfy the $(F_{SH}2)$ condition since $1|(x|x) = x \in U$ with $x|(1|1) = 1$, but $(x|(z|z))|((1|(z|z))|(1|(z|z))) = (x|(z|z))|(z|z) = (x|t)|t = u \notin U$. Therefore, it is not a filter.

(ii) $(\Rightarrow:)$ We suppose that F verifies $(F_{SH}1)$. For each $a, b \in F$, by Definition 2.2, $(S2)$ and $(S3)$, and also Lemma 2.6 (i), we obtain that

$$\begin{aligned}
&a|((b|(((a|b)|(a|b))|((a|b)|(a|b))))|(b|(((a|b)|(a|b))|((a|b)|(a|b)))))\\
&= a|((b|(a|b))|(b|(a|b)))\\
&= a|(((a|b)|b)|((a|b)|b))\\
&= ((a|b)|(a|b))|(a|b) = 1.
\end{aligned}$$

Hence $a \circledast b \in F$, showing that F is an order-filter.
$(\Leftarrow:)$ Let F be an order-filter closed under the operation $\circledast$. If $a, a|(b|b) \in F$, then $a \circledast a|(b|b) \in F$. By using the commutativity of Sheffer stroke operation, $(S2)$ in Definition 2.2 and Corollary 2.9, we have

$$\begin{aligned}
a \circledast a|(b|b) &= ((a|(b|b))|a)|((a|(b|b))|a)\\
&= (((b|b)|((a|a)|(a|a)))|((a|a)|(a|a)))|(((b|b)|((a|a)|(a|a)))|((a|a)|(a|a)))\\
&= \neg(\neg b \vee \neg a)\\
&= a \wedge b,
\end{aligned}$$

which proves that $a \wedge b \in F$, that is $b \in F$. Therefore, $(F_{SH}1)$ is satisfied.

(iii) Let $\mathcal{A}$ be associative. Then, the conditions $(F_{SH}2)$ and $(F_{SH}3)$ are satisfied, since the following inequalites hold in $\mathcal{A}$

$$\begin{aligned} z|(x|x) \leq 1 \quad &\Rightarrow \quad 1|(y|(z|z)) \leq (z|(x|x))|(y|(z|z)) \\ &\Rightarrow \quad (y|(z|z))|(y|(z|z)) \leq (z|(x|x))|(y|(((z|z)|(z|z))|z)) \\ &\Rightarrow \quad ((z|z|(z|z))|(y|y)) \leq (z|(x|x))|(y|(((z|z)|(z|z))|z)) \\ &\Rightarrow \quad z|(y|y) \leq (y|(z|z))|((x|(z|z))|(x|(z|z))) \end{aligned}$$

and

$$\begin{aligned} x|x \leq 1 \quad &\Rightarrow \quad ((x|x)|y)|((x|x)|y) \leq (x|x)|((x|x)|y) \\ &\Rightarrow \quad x|(y|y) \leq x|y \\ &\Rightarrow \quad x|(y|y) \leq (1|(x|x))|y \\ &\Rightarrow \quad x|(y|y) \leq ((z|(z|z))|x)|((y|y)|(y|y)) \\ &\Rightarrow \quad x|(y|y) \leq (z|(x|x))|((z|(y|y))|(z|(y|y))). \end{aligned}$$

Therefore, if $\mathcal{A}$ is an associative Sheffer stroke basic algebra, then the compatible filters are precisely subsets of $\mathcal{A}$ verifying $(F_{SH}1)$. $\square$

Definition 3.8. *Let Θ be an equivalence relation on A. If it satisfies the following condition*

$$(a, b) \in \Theta \Rightarrow (a|(c|c), b|(c|c)) \in \Theta,$$

for all $a, b, c \in A$, then it is called the right congruence of $\mathcal{A}$ and the set of the right congruences of $\mathcal{A}$ is denoted by $\mathbf{Con}_R(\mathcal{A})$.

If it satisfies the following condition for all $a, b, c \in A$

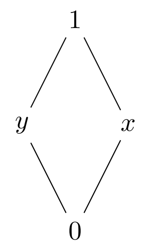
$$(a, b) \in \Theta \Rightarrow ((a|a)|c, (b|b)|c) \in \Theta,$$

then it is called the left congruence of $\mathcal{A}$ and the set of the right congruences of $\mathcal{A}$ is denoted by $\mathbf{Con}_L(\mathcal{A})$.

If $\Theta \in \mathbf{Con}_R(\mathcal{A})$ and $\Theta \in \mathbf{Con}_L(\mathcal{A})$, then the Θ is called the congruence of $\mathcal{A}$ and the set of the congruences of $\mathcal{A}$ is denoted by $\mathbf{Con}(\mathcal{A})$, namely

$$\mathbf{Con}(\mathcal{A}) = \mathbf{Con}_R(\mathcal{A}) \cap \mathbf{Con}_L(\mathcal{A}).$$

Example 3.9. *Let $A = \{0, x, y, 1\}$, where $0 < x, y < 1$ but a and b are not comparable. The relations of elements in A are given and the operation $|$ on this structure is defined as the following figure and the table, respectively:*



$\mid$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	1

Then $\mathcal{A} = (A; |)$ is a Sheffer stroke basic algebra. If the relation Θ_1 is defined on $\mathcal{A}$ as

$$(a, b) \in \Theta_1 \quad \textit{if and only if} \quad a|b = 0$$

for all $a, b \in A$, then $\Theta_1 \in \mathbf{Con}_L(\mathcal{A})$ but $\Theta_1 \notin \mathbf{Con}_R(\mathcal{A})$.

Example 3.10. *Let $\mathcal{A} = (A; |)$ be a Sheffer stroke basic algebra given in Example 3.2. The relation Θ_2 is defined on $\mathcal{A}$ by*

$$(a, b) \in \Theta_2 \quad \textit{if and only if} \quad (a|(b|b))|(b|b) = 1 \quad \textit{and} \quad ((a|a)|(b|b))|(b|b) = 0$$

for all $a, b \in A$, then $\Theta_2 \in \mathbf{Con}_L(\mathcal{A})$ and $\Theta_2 \in \mathbf{Con}_R(\mathcal{A})$. So, $\Theta_2 \in \mathbf{Con}(\mathcal{A})$.

For this example, $\mathcal{A}/\Theta_2 = \{\{0, 1\}, \{x, t\}, \{y, z\}\}$ is a partition of $\mathcal{A}$.

Lemma 3.11. *The following biconditional statement*

$$(a, b) \in \Theta \Leftrightarrow (a|a, b|b) \in \Theta$$

is satisfied for each $\Theta \in \mathbf{Con}_R(\mathcal{A})$.

Lemma 3.12. *If the identity $a|(b|b) = b|(a|a)$ is satisfied for each $a, b \in A$, then $\mathbf{Con}_R(\mathcal{A}) = \mathbf{Con}(\mathcal{A})$.*

Proof. Assume that $\Theta \in \mathbf{Con}_R(\mathcal{A})$ and $(a, b) \in \Theta$. Then $(a|a, b|b) \in \Theta$. For each $c \in A$

$$c|(a|a) = a|(c|c)\Theta b|(c|c) = c|(b|b) \Rightarrow \Theta \in \mathbf{Con}_L(\mathcal{A}).$$

Therefore, $\mathbf{Con}_R(\mathcal{A}) = \mathbf{Con}(\mathcal{A})$. $\square$

Lemma 3.13. *Let F be a filter of $\mathcal{A}$. The relation $\Theta(F)$ on $\mathcal{A}$ is given by*

$$(a,b) \in \Theta(F) \quad \textit{if and only if} \quad a|(b|b) \in F \textit{ and } b|(a|a) \in F.$$

Then, we have the following conclusions:

(*i*) $\Theta(F) \in \mathbf{Con}_R(\mathcal{A})$ *with* $[1]_{\Theta(F)} = F$,

(*ii*) *If* $F \in \mathbf{CF}(\mathcal{A})$, *then* $\Theta(F) \in \mathbf{Con}(\mathcal{A})$.

Proof. (*i*) Assume that $(a,b),(b,c) \in \Theta(F)$ where $a \leq b$ and $b \leq c$. Then we have $a|(b|b)$, $b|(a|a) \in F$ and $a|(b|b) = 1$, and also $b|(c|c), c|(b|b) \in F$ and $b|(c|c) = 1$. By using $(F_{SH}2)$, we get $(b|(a|a))|((c|(a|a))|(c|(a|a))) \in F$, since $a|(b|b) = 1$ and $b|(c|c) \in F$. From $(F_{SH}1)$, we have $c|(a|a) \in F$. Similarly, it is easy to get $a|(c|c) \in F$. Therefore, $(a,c) \in \Theta(F)$; thus $\Theta(F)$ is transitive. Moreover, from Lemma 2.6 (i), we have $x|(x|x) = 1 \in F$ for all $x \in A$. Therefore, $\Theta(F)$ is reflexive. $\Theta(F)$ is symmetric by its definition. Hence, $\Theta(F)$ is an equivalence relation on A.

Let $(a,b) \in \Theta(F)$. Then we have $a|(b|b), b|(a|a) \in F$. If $a|(b|b) \in F$, then by $(F_{SH}2)$, we get

$$(b|(c|c))|((a|(c|c))|(a|(c|c))) \in F. \tag{1}$$

If $b|(a|a) \in F$, then in a similar way we obtain that

$$(a|(c|c))|((b|(c|c))|(b|(c|c))) \in F. \tag{2}$$

From (1) and (2), we conclude that $(a|(c|c), b|(c|c)) \in \Theta(F)$. Therefore, $\Theta(F) \in \mathbf{Con}_R(\mathcal{A})$.

Let $(a,1) \in \Theta(F)$. Then $1|(a|a) = a \in F$. If $a \in F$, then by Lemma 2.6 (iii), we have $a = 1|(a|a) \in F$ and by Lemma 2.6 (ii), we get also $1 = a|(1|1)$, since $1 \in F$. Therefore, $(a,1) \in \Theta(F)$. As a result, $[1]_{\Theta(F)} = \{a : (a,1) \in \Theta(F)\} = F$.

(*ii*) Let $F \in \mathbf{CF}(\mathcal{A})$. Assume that $(a,b) \in \Theta(F)$. Therefore, $a|(b|b), b|(a|a) \in F$. By $(F_{SH}3)$, we have that

$$(c|(a|a))|((c|(b|b))|(c|(b|b))) \in F \tag{3}$$

and

$$(c|(b|b))|((c|(a|a))|(c|(a|a))) \in F. \tag{4}$$

From (3) and (4), we conclude that $\Theta(F) \in \mathbf{Con}(\mathcal{A})$. $\square$

Example 3.14. *Assume that* $\mathcal{A} = (A; |)$ *is a Sheffer stroke basic algebra in Example 3.2. We have shown that* $F = \{x, z, 1\} \in \mathbf{CF}(\mathcal{A})$ *was a compatible filter. Therefore,* $\Theta(F) \in \mathbf{Con}(\mathcal{A})$ *with partitions* $\{0, y, t\}$ *and* $\{x, z, 1\}$*. So,* $\mathcal{A}/\Theta(F) = \{\{x, z, 1\}, \{0, y, t\}\}$.

Assume that $\mathcal{B} = (B; |)$ *is a Sheffer stroke basic algebra in Example 3.3. We have* $Q = \{1, t\} \notin \mathbf{CF}(\mathcal{A})$*. Thereby,* $\Theta(Q) \notin \mathbf{Con}_R(\mathcal{A})$ *because* $(t, 1) \in \Theta(Q)$ *but* $(1|(x|x),\ t|(x|x)) = (x, u) \notin \Theta(Q)$.

Definition 3.15. *Let* $\Theta(F)$ *be a congruence of* $\mathcal{A}$*. Then the operation* $|$ *is defined on* $\mathcal{A}/\Theta(F)$ *as*

$$[a]_{\Theta(F)} \mid [b]_{\Theta(F)} := [a|b]_{\Theta(F)}.$$

Theorem 3.16. *If* F *is a compatible filter of* $\mathcal{A}$*, then* $\mathcal{A}/\Theta(F)$ *is a Sheffer stroke basic algebra.*

Proof. Assume that F is compatible filter. By Lemma 3.13, we have $\Theta(F) \in \mathbf{Con}(\mathcal{A})$. Then, we can provide the identities $(SH1) - (SH3)$ as follows:

$(SH1)$

$$\begin{aligned} ([a]_{\Theta(F)}|([a]_{\Theta(F)}|[a]_{\Theta(F)}))|([a]_{\Theta(F)}|[a]_{\Theta(F)}) &= [(a|(a|a))|(a|a)]_{\Theta(F)} \\ &= [a]_{\Theta(F)}. \end{aligned}$$

$(SH2)$

$$\begin{aligned} & ([a]_{\Theta(F)}|([b]_{\Theta(F)}|[b]_{\Theta(F)}))|([b]_{\Theta(F)}|[b]_{\Theta(F)}) \\ = \ & [(a|(b|b))|(b|b)]_{\Theta(F)} \\ = \ & [(b|(a|a))|(a|a)]_{\Theta(F)} \\ = \ & ([b]_{\Theta(F)}|([a]_{\Theta(F)}|[a]_{\Theta(F)}))|([a]_{\Theta(F)}|[a]_{\Theta(F)}). \end{aligned}$$

$(SH3)$

$$\begin{aligned} & ((([a]_{\Theta(F)}|([b]_{\Theta(F)}|[b]_{\Theta(F)}))|([b]_{\Theta(F)}|[b]_{\Theta(F)}))|([c]_{\Theta(F)}|[c]_{\Theta(F)})) \\ & |(([a]_{\Theta(F)}|([c]_{\Theta(F)}|[c]_{\Theta(F)}))|([a]_{\Theta(F)}|([c]_{\Theta(F)}|[c]_{\Theta(F)}))) \\ = \ & [(((a|(b|b))|(b|b))|(c|c))|((a|(c|c))|(a|(c|c)))]_{\Theta(F)} \\ = \ & [a|(a|a)]_{\Theta(F)} \\ = \ & [a]_{\Theta(F)}|([a]_{\Theta(F)}|[a]_{\Theta(F)}). \end{aligned}$$

□

Definition 3.17. *A filter* F *of* $\mathcal{A}$ *is called a prime filter if it satisfies the condition*

$$a|(b|b) \in F \ \text{or} \ \ b|(a|a) \in F$$

for each $a, b \in A$.

Example 3.18. *Let* $\mathcal{B} = (B; |)$ *be a Sheffer stroke basic algebra given in Example 3.3. For each* $a, b \in B$, $a|(b|b) \in F$ *or* $b|(a|a) \in F$ *are satisfied, where* $F = \{t, u, v, 1\}$. *Hence,* $F = \{t, u, v, 1\}$ *is a prime filter of* $\mathcal{B}$.

Lemma 3.19. *Let* F *be a filter of* $\mathcal{A}$. *Then we have the following biconditional statement*

$$a|(b|b) \in F \ \text{if and only if} \ \ b \in F$$

for $a, b \in A$.

Proof. ($\Rightarrow$:) We assume that $a|(b|b) \in F$. From Lemma 3.2 in [18], we get

$$(b|(a|(b|b)))|(a|(b|b)) = 1.$$

By using Lemma 2.7 (ii) in [18], we obtain $(b|(a|(b|b))) = b|b$, thus

$$(b|b)|(a|(b|b)) = 1.$$

From the commutativity of Sheffer stroke operation, we have

$$(a|(b|b))|(b|b) = 1.$$

By the definition of $\leq$ on $\mathcal{A}$, we have

$$a|(b|b) \leq b.$$

So, $b \in F$.
($\Leftarrow$:) Assume that $b \in F$. Then, for each $a \in A$,

$$\begin{aligned} a \leq 1 \quad &\Rightarrow \quad 1|(b|b) \leq a|(b|b) \qquad \text{(by using Lemma 2.4)} \\ &\Rightarrow \quad b \leq a|(b|b) \qquad\qquad \text{(by using Lemma 2.6).} \end{aligned}$$

Therefore $a|(b|b) \in F$. □

Remark 3.20. *Let* F *be a filter of* $\mathcal{A}$. *If* $b \in F$, *then*

$$a|(b|b) = b$$

for each $a \in \mathcal{A}$.

Lemma 3.21. *[18] The operations* $\neg$ *and* $\oplus$ *are defined as follows*

$$\neg a := a|a \quad \text{and} \quad a \oplus b := (a|a)|(b|b).$$

Then $\mathcal{A} = (A; \oplus, \neg, 0, 1)$ *is a basic algebra.*

Definition 3.22. *Let* $\Theta(F)$ *be a congruence of* $\mathcal{A}$*. Then the operation* $\oplus$ *is defined on* $\mathcal{A}/\Theta(F)$ *as follows:*

$$[a]_{\Theta(F)} \ \oplus \ [b]_{\Theta(F)} := [a \oplus b]_{\Theta(F)}.$$

As a result of Theorem 3.16 and Definition 3.22, we obtain the following corollary.

Corollary 3.23. *If* F *is a compatible filter of* $\mathcal{A}$*, then* $(\mathcal{A}/\Theta(F); \oplus, \neg, 0, 1)$ *is a basic algebra.*

Remark 3.24. *It can be said from Lemma 3.7 (i) that every filter is an order-filter but the converse of that statement is not necessarily true. It can also be said from Definition 3.1 that every compatible filter is a filter but the converse of its is not true in general because of the Example 3.3. In addition to these, every prime filter is a filter from Definition 3.17, but the converse is not true because* $F = \{1, t\}$ *is a filter in Example 3.3 but* $x|(y|y) = x|u = v \notin F$ *and* $y|(x|x) = y|v = u \notin F$*. Therefore,* F *is not a prime filter in this example.*

By the help of Remark 3.24, the relationship between compatible filters, prime filters, filters and order-filters in Sheffer stroke basic algebras could be explained as the following diagram:

$$\begin{tikzcd}
\text{Prime Filter} \arrow[dr] & \text{Compatible Filter} \arrow[d] \\
& \text{Filter} \arrow[d] \\
& \text{Order-Filter}
\end{tikzcd}$$

Theorem 3.25. *If* A *is a Sheffer stroke basic algebra with* $a|(b|b) = b|(a|a)$ *for each element in* A*, then the above diagram becomes complete.*

Proof. Assume that A be a Sheffer stroke basic algebra with $a|(b|b) = b|(a|a)$ for every element in A. Then we can give other directions of arrows in the above diagram.

(Order-Filter)⇒(Filter): Let U be a order-filter and $a \in U$. Since $a \in U$ and $a|(1|1) = 1$, then $1 \in U$. Now, we can verify the conditions $(F_{SH}1)$ and $(F_{SH}2)$. By using the hypothesis, $1 \in U$ and $1|(b|b) = b|(1|1) = 1$, then $b \in U$. So, the condition $(F_{SH}1)$ holds. If $a|(b|b) \in U$ and $b|(a|a) = 1$, since $a|(b|b) = b|(a|a) = 1$, then $a \leq b$ and $b \leq a$. Therefore, $a = b$. So, we obtain $(b|(c|c))|((a|(c|c))|(a|(c|c))) = (a|(c|c))|((a|(c|c))|(a|(c|c))) = 1 \in U$, i.e., U satisfies the condition $(F_{SH}2)$. Then, U is an filter.

(Filter)⇒(Compatible Filter): It is clearly obtained by using the hypothesis and the condition $(F_{SH}2)$.

(Filter)⇒(Prime Filter): Assume that F be a filter. From the hypothesis, we get $1 = 0|(a|a) = a|(0|0) = a|a$ for each $a \in A$. By using this equality, for each $a, b \in F$, $b|(a|a) = b|1 = b|b = 1 \in F$. Therefore, F is a prime filter. □

Theorem 3.26. *F is a prime filter if and only if $\mathcal{A}/\Theta(F)$ is linearly ordered.*

Proof. (⇒:) Let F be a prime filter. Then we have $a|(b|b) \in F$ or $b|(a|a) \in F$ for each $a, b \in A$. Assume that $a|(b|b) \in F$. So, $b \in F$ is obtained from Lemma 3.19. If $x \in [a]_{\Theta(F)}$, we get $a|(x|x) \in F$ and $x|(a|a) \in F$. By using Lemma 3.19, we obtain $x \in F$ and $a \in F$. Since $b \in F$ and $x \in F$, we have $b = x|(b|b) \in F$ and $x = b|(x|x) \in F$ from Lemma 3.19. Hence $x \in [b]_{\Theta(F)}$. Therefore $[a]_{\Theta(F)} \subseteq [b]_{\Theta(F)}$, i.e., $[a]_{\Theta(F)} \leq [b]_{\Theta(F)}$. Similarly, one can show that $b|(a|a) \in F$.
(⇐:) Assume that $\mathcal{A}/\Theta(F)$ is linearly ordered. Then we have $[a]_{\Theta(F)} \leq [b]_{\Theta(F)}$ or $[b]_{\Theta(F)} \leq [a]_{\Theta(F)}$. If $[a]_{\Theta(F)} \leq [b]_{\Theta(F)}$, then $a \in [a]_{\Theta(F)}$ and $a \in [b]_{\Theta(F)}$. So, $a|(b|b) \in F$ and $b|(a|a) \in F$. □

Lemma 3.27. *Let F be a prime filter. Then, the following conditions are equivalent:*

(*i*) *$a \vee b = 1$ and also $a \in F$ or $b \in F$ for $a, b \in A$.*

(*ii*) *$a|(b|b) \in F$ or $b|(a|a) \in F$ for $a, b \in A$.*

(*iii*) *$(a|(b|b))|(b|b) \in F$ and also $a \in F$ or $b \in F$ for $a, b \in A$.*

(*iv*) *$\mathcal{A}/\Theta(F)$ is linearly ordered.*

Proof. $(i) \Rightarrow (ii)$ Assume that $a \vee b = 1$ and also $a \in F$ or $b \in F$ for each $a, b \in A$. From Lemma 3.13 (ii) in [18], $a \vee b = 1$ if and only if $a|(b|b) = b$ or $b|(a|a) = a$. So, $a|(b|b) \in F$ or $b|(a|a) \in F$.

$(ii) \Rightarrow (iii)$ Assume that $a|(b|b) \in F$ or $b|(a|a) \in F$ for $a, b \in A$ and $(a|(b|b))|(b|b) \in F$. By using Lemma 3.19, $a = b|(a|a) \in F$ or $b = a|(b|b) \in F$.

$(iii) \Rightarrow (iv)$ Suppose that $a \in F$. For all $b \in A$, $b \leq 1$. So, $1|(a|a) \leq b|(a|a)$. Then we get $a \leq b|(a|a)$. Therefore $b|(a|a) \in F$. If $b \in F$, then we similarly obtain $a|(b|b) \in F$. By using Theorem 3.26, we conclude that $\mathcal{A}/\Theta(F)$ is linearly ordered.

$(iv) \Rightarrow (i)$ We suppose that $\mathcal{A}/\Theta(F)$ is linearly ordered. By Theorem 3.26, $a|(b|b) \in F$ or $b|(a|a) \in F$. Assume that $a|(b|b) \in F$. Then by using Lemma 3.19, $b = a|(b|b) \in F$. So, we obtain $1 = (b|b)|b = (a|(b|b))|(b|b) = a \vee b$. $\square$

As a result, we conclude that the following theorem.

Theorem 3.28. *A filter F of $\mathcal{A}$ is prime if and only if it satisfied at least one condition in Lemma 3.27.*

Lemma 3.29. *Let $F \in \mathbf{CF}(\mathcal{A})$. Then the following statements are equivalent for each $a, b \in A$:*

(i) $(a, b) \in \Theta(F)$.

(ii) *There is an element $c \in A$ where $a, b \leq c$ such that $(a, c), (b, c) \in \Theta(F)$.*

(iii) *There are $f, g \in F$ such that $f|(a|a) = g|(b|b)$ and $a \leq f, b \leq g$.*

(iv) *There are $f, g \in F$ such that $f \circledast a = g \circledast b$ and $a|a \leq f, b|b \leq g$.*

Proof. $(i) \Rightarrow (ii)$ We assume that $(a, b) \in \Theta(F)$ and there exists $c \in A$ where $a, b \leq c$. So, by the definition of $\Theta(F)$, $a|(b|b), b|(a|a) \in F$ and also by the definition of $\leq$, $a|(c|c) = b|(c|c) = 1 \in F$. If we choose $c = a \vee b = ((a|(b|b))|(b|b))$, then from Lemma 2.6 (iv), $c|(b|b) = ((a|(b|b))|(b|b))|(b|b) = a|(b|b) \in \Theta(F)$. Therefore, $(b, c) \in \Theta(F)$. Since $(a|(b|b))|(b|b) = (b|(a|a))|(a|a)$, we analogously get $c|(a|a) \in \Theta(F)$. So $(a, c) \in \Theta(F)$.

$(ii) \Rightarrow (iii)$ We assume that $f = b|(a|a)$ and $g = (a|(b|b))$. Then by using Lemma 2.6 (iv), we obtain $f \vee a = ((b|(a|a))|(a|a))|(a|a) = b|(a|a) = f$ and $g \vee b = ((a|(b|b))|\ (b|b))|(b|b) = (a|(b|b)) = g$. Therefore, $a \leq f$ and $b \leq g$. In addition to this, by using $(SH2)$, we have $f|(a|a) = (b|(a|a))|(a|a) = (a|(b|b))|(b|b) = g|(b|b)$.

$(iii) \Rightarrow (iv)$ If we take $f = b|(a|a)$ and $g = a|(b|b)$, then we have $f|(a|a) = g|(b|b) = a \vee b$. Since $b \leq f|(a|a)$, we obtain $b|(a|a) \geq f|((a|a)|(a|a)) = f \vee a = f$ by using Lemma 2.4 (iii). As $f \in F$, we conclude $b|(a|a) \in F$. In a similar way, we reach $a|(b|b) \in F$. Hence $(a, b) \in \Theta(F)$ from Lemma 2.5. And also, we get $(a|a, b|b) \in \Theta(F)$ from Lemma 3.11. Therefore, there are $f', g' \in F$ such that $a|a \leq f'$ and $b|b \leq g'$, and also we have $f'|((a|a)|(a|a)) = g'|((b|b)|(b|b))$. By using Definition 2.2 $(S2)$, we obtain $f'|a = g'|b$. Then we get easily $f' \circledast a = g' \circledast b$.

$(iv) \Rightarrow (i)$ We assume that there are $f, g \in F$ such that $a|a \leq f$, $b|b \leq g$ and $f' \circledast a = g' \circledast b$. As $\Theta(F) \in \mathbf{Con}_R(\mathcal{A})$, then we get $(a|a, b|b) \in \Theta(F)$. By using Lemma 3.11, we obtain $(a, b) \in \Theta(F)$. □

Theorem 3.30. *Let* $\Theta \in \mathbf{Con}(\mathcal{A})$. *Then*

- (i) $[1]_\Theta$ *is a compatible filter of* $\mathcal{A}$.
- (ii) $\Theta([1]_\Theta) = \Theta$.

Proof. (i) Let $[1]_\Theta = F$. Consider $[1]_\Theta$ where $\Theta \in \mathbf{Con}(\mathcal{A})$.
• Assume that $a \in F$ and $a|(b|b) \in F$. Then, we get $(a, 1) \in \Theta$ and $(a|(b|b), 1) \in \Theta$ since $a|(1|1) = 1$ and $(a|(b|b))|(1|1) = 1$. Also,

$$(a|((a|(b|b))|(a|(b|b))), 1) = (a|((a|(b|b))|(a|(b|b))), 1|(1|1)) \in \Theta$$

and hence

$$\begin{array}{rcl} b = 1|(b|b) & \Theta & (a|((a|(b|b))|(a|(b|b))))|(b|b) \\ & \Theta & (1|((1|(b|b))|(1|(b|b))))|(b|b) = b|(b|b) = 1 \\ & \Rightarrow & (b, 1) \in \Theta \\ & \Rightarrow & b \in F. \end{array}$$

Thus, $(F_{SH}1)$ holds.
• Assume that $a|(b|b) \in F$. Therefore $(a|(b|b), 1) \in \Theta$. By using Definition 2.2 $(S3)$, we obtain

$$\begin{array}{rcl} (b|(c|c))|((a|(c|c))|(a|(c|c))) & = & ((1|(b|b))|(c|c))|((a|(c|c))|(a|(c|c))) \\ & \Theta & (((a|(b|b))|(b|b))|(c|c))|((a|(c|c))|(a|(c|c))) = 1 \\ & \Rightarrow & (b|(c|c))|((a|(c|c))|(a|(c|c))) \in F \end{array}$$

for all $c \in A$. Thus, $(F_{SH}2)$ is satisfied.
• Assume that $a|(b|b) \in F$ and $b|(a|a) \in F$. Then $(a|(b|b), 1) \in \Theta$ and

$(b|(a|a), 1) \in \Theta$. So,

$$\begin{aligned} 1 &= (c|(a|a))|((c|(a|a))|(c|(a|a))) \\ &= (c|(a|a))|((c|((1|(a|a))|(1|(a|a))))|(c|((1|(a|a))|(1|(a|a))))). \end{aligned}$$

Since $b|(a|a)\Theta 1$, we can write $b|(a|a)$ instead of 1 in the last part of above equality. Then

$$(c|(a|a))|((c|((1|(a|a))|(1|(a|a))))|(c|((1|(a|a))|(1|(a|a)))))\Theta(c|(a|a))|((c|(((b| (a|a))|(a|a))|((b|(a|a))|(a|a))))|(c|(((b|(a|a))|(a|a))|((b|(a|a))|(a|a))))).$$

By using $(SH2)$,

$$(c|(a|a))|((c|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b))))|(c|(((a|(b|b))|(b|b))|((a|(b|b)) |(b|b))))).$$

As $a|(b|b)\Theta 1$, we can write 1 instead of $a|(b|b)$ in the last equation. Then, we obtain

$$(c|(a|a))|((c|((1|(b|b))|(1|(b|b))))|(c|((1|(b|b))|(1|(b|b))))).$$

By using the equility of $1 = 1|(b|b)$ which is obtained by Lemma 2.6 (ii), we conclude

$$(c|(a|a))|((c|(b|b))|(c|(b|b))).$$

This shows that $(c|(a|a))|((c|(b|b))|(c|(b|b))) \in F$. Therefore, $(F_{SH}3)$ is satisfied. So, $[1]_\Theta$ is a compatible filter of $\mathcal{A}$.

(ii) Let $(a, b) \in \Theta([1]_\Theta)$. Then, from Lemma 3.29, there are $f, g \in [1]_\Theta$ such that $f|(a|a) = g|(b|b)$. We can deduce

$$a = 1|(a|a)\Theta f|(a|a) = g|(b|b)\Theta 1|(b|b) = b$$

and so $(a, b) \in \Theta$. This means that $\Theta \subseteq \Theta([1]_\Theta)$. On the other side, let $(a, b) \in \Theta$ and also, $(a|(b|b), 1) \in \Theta$ and $(b|(a|a), 1) \in \Theta$. Therefore, $a|(b|b) \in [1]_\Theta$ and $b|(a|a) \in [1]_\Theta$. Hence, $(a, b) \in \Theta([1]_\Theta)$. This means that $\Theta([1]_\Theta) \subseteq \Theta$. Consequently, $\Theta([1]_\Theta) = \Theta$. □

For a representation theorem of Sheffer stroke basic algebra, we need to construct lattice structure of Sheffer stroke basic algebra.

Let $\mathcal{A}_\mathcal{L} = (A_\mathcal{L}; |, \vee, \wedge, 0, 1)$ be a lattice Sheffer stroke basic algebra.

Definition 3.31. *A lattice Sheffer stroke basic algebra $\mathcal{A}_{\mathcal{L}}$ is called a lattice $\mathcal{H}$ Sheffer stroke basic algebra, if the identity*

$$a \vee b \vee ((a \wedge b)|(c|c)) = 1$$

is satisfied for each $a, b, c \in A_{\mathcal{L}}$.

Proposition 3.32. *Let $\mathcal{A}_{\mathcal{L}}$ be a lattice Sheffer stroke basic algebra and F be a prime filter of $\mathcal{A}$. Then $(\mathcal{A}/\Theta(F))_{\mathcal{L}}$ is a lattice $\mathcal{H}$ Sheffer stroke basic algebra.*

Proof. It follows from Lemma 3.27. □

Definition 3.33. *Let $\mathcal{A}_{\mathcal{L}_1}$ be a lattice Sheffer stroke basic algebra and $\mathcal{A}_{\mathcal{L}_2}$ be a lattice $\mathcal{H}$ Sheffer stroke basic algebra. Then a lattice Sheffer stroke epimorphism $f_\alpha : \mathcal{A}_{\mathcal{L}_1} \to \mathcal{A}_{\mathcal{L}_2}$ is called a representation of $\mathcal{A}_{\mathcal{L}_1}$ on $\mathcal{A}_{\mathcal{L}_2}$.*

Lemma 3.34. *Let $\mathcal{A}_{\mathcal{L}}$ be a lattice Sheffer stroke basic algebra and F be a prime filter of $\mathcal{A}$. Then the mapping*

$$\begin{array}{rcl} f_\alpha : \mathcal{A}_{\mathcal{L}} & \longrightarrow & (\mathcal{A}/\Theta(F))_{\mathcal{L}} \\ x & \longmapsto & ([x]_{\Theta(F)})_{\mathcal{L}} \end{array}$$

is a lattice Sheffer stroke epimorphism.

Theorem 3.35. *Let $\mathcal{A}_{\mathcal{L}}$ be a lattice Sheffer stroke basic algebra and F be a prime filter of $\mathcal{A}$. Then we have a representation of $\mathcal{A}_{\mathcal{L}}$ on $(\mathcal{A}/\Theta(F))_{\mathcal{L}}$.*

Proof. It comes from Definition 3.33 and Lemma 3.34. □

4 Conclusion

Sheffer stroke basic algebra corresponds to the reduction of basic algebra which consists of only one operation and it represents the algebraic counterpart of different non-classical logics because it does not only include orthomodular lattices but also yields an axiomatization of the logic of quantum mechanics along with MV-algebras which is obtained an axiomatization of many-valued Łukasiewicz logics. In this paper, we define filter, compatible filter and congruence notions for Sheffer stroke basic algebra, and we explain these notions by examples. In the last part of this study, constructing a bridge between filter and congruence on this structure, we prove that each congruence of this structure is uniquely determined by its filter $[1]_\Theta$ and so to characterize congruences, it will be adequate to describe filters. Finally, we give a representation theorem of Sheffer stroke basic algebra.

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